

MTH 234 - Quiz 2

Due 29 May at the beginning of class

Name: Solutions.

You may work together on solving these problems, but what you hand in must be your work written in your own words.

1. (5 points) If \vec{r} is a twice-differentiable vector-valued function, show that

$$\frac{d}{dt} [\vec{r}(t) \times \vec{r}'(t)] = \vec{r}(t) \times \vec{r}''(t)$$

By the product rule,

$$(\vec{r}(t) \times \vec{r}'(t))' = \underbrace{\vec{r}'(t) \times \vec{r}'(t)}_{=0} + \vec{r}(t) \times \vec{r}''(t)$$

But $\vec{r}'(t) \times \vec{r}'(t) = \vec{0}$ for all t ;

this is true for any vector. So this proves the claim. \square

2. (5 points) If $\vec{r}(t) \neq 0$, show that

$$\frac{d}{dt} |\vec{r}(t)| = \frac{1}{|\vec{r}(t)|} \vec{r}(t) \cdot \vec{r}'(t)$$

Hint: One way is to write $|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$ and differentiate both sides with the chain rule.

By the chain rule,

$$\frac{d}{dt} |\vec{r}(t)|^2 = 2|\vec{r}(t)| \frac{d}{dt} |\vec{r}(t)|$$

$$\begin{aligned} \text{Also, } \frac{d}{dt} \vec{r}(t) \cdot \vec{r}(t) &= \vec{r}(t) \cdot \vec{r}'(t) + \vec{r}'(t) \cdot \vec{r}(t) \\ &= 2\vec{r}(t) \cdot \vec{r}'(t). \end{aligned}$$

$$\text{Recall } \kappa = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}.$$

3. (5 points) Consider the function $\vec{r}(t) = \langle t, \ln t, t \rangle$, $t > 0$.

(a) Compute the curvature κ of \vec{r} .

(b) What is $\lim_{t \rightarrow \infty} \kappa(t)$? Describe this geometrically.

$$\text{Here, } \vec{r}' = \langle 1, \frac{1}{t}, 1 \rangle$$

$$\vec{r}'' = \langle 0, -\frac{1}{t^2}, 0 \rangle.$$

$$a) \kappa = \frac{|\langle 1, \frac{1}{t}, 1 \rangle \times \langle 0, -\frac{1}{t^2}, 0 \rangle|}{\|\langle 1, \frac{1}{t}, 1 \rangle\|^3}$$

$$\text{Cross product: } \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \frac{1}{t} & 1 \\ 0 & -\frac{1}{t^2} & 0 \end{vmatrix} = +\frac{1}{t^2} \vec{i} - \frac{1}{t^2} \vec{k}$$

$$\therefore \kappa = \frac{\sqrt{\frac{1}{t^4} + \frac{1}{t^4}}}{(\sqrt{t^2 + \frac{1}{t^2}})^3} = \frac{\sqrt{2}/t^2}{(\sqrt{t^2 + 1}/t)^3} \Rightarrow \boxed{\kappa = \frac{\sqrt{2} t}{(\sqrt{t^2 + 1})^3}}$$

b) Use L'Hopital or notice that $\kappa \leq \frac{\sqrt{2} t}{t^{3/2} + t^3} \leq \frac{2}{t^2} \rightarrow 0$

to get $\boxed{\lim_{t \rightarrow \infty} \kappa(t) = 0}$. Geometrically, the curve is straightening

4. (5 points) A particle moves along the helix $x(t) = 3 \cos t$, $y(t) = 3 \sin t$; $z(t) = t$, starting at time 0 (at the point $(3, 0, 0)$). How long does it take for the particle to travel 1 unit?

as t increases,

$$\text{Arc length: } s(t) = \int_0^t \|\langle x'(u), y'(u), z'(u) \rangle\| du$$

$$= \int_0^t \|\langle -3 \sin u, 3 \cos u, 1 \rangle\| du$$

$$= \int_0^t \sqrt{(-3 \sin u)^2 + (3 \cos u)^2 + 1} du$$

$$= \int_0^t \sqrt{10} du = \sqrt{10} t.$$

$$\therefore s(t) = \sqrt{10} t.$$

$$\text{Set } s(t) = 1 : 1 = \sqrt{10} t \Rightarrow$$

$$\boxed{t = \frac{1}{\sqrt{10}} \text{ time units}}.$$